

The Corruption Bound, Log Rank, and Communication Complexity

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Abstract

We prove that for every sign matrix A there is a deterministic communication protocol that uses $O(\text{corr}_{1/4}(A) \log^2 \text{rank}(A))$ bits of communication, where $\text{corr}_{1/4}(A)$ is the corruption/rectangle bound with error $1/4$. This bound generalizes several of the known upper bounds on deterministic communication complexity, involving nondeterministic complexity, randomized complexity, information complexity notions, and rank.

It also implies that the corruption bound is a lower bound on exact quantum communication complexity, if and only if quantum communication is polynomially equivalent to deterministic communication complexity.

Our bounds are in the spirit of the results of Gavinsky and Lovett [5]. We also give a simple proof for the new upper bound on communication complexity in terms of rank proved by Lovett [15].

1 Introduction

The communication complexity literature is mainly concerned with proving lower bounds, and indeed communication complexity lower bounds are used in various areas of theoretical computer science. Nonetheless, upper bounds are also very interesting. Among the most intriguing upper bounds are those expressed in terms of known lower bounds. Let A be a sign matrix, and denote by $D(A)$ the deterministic communication complexity of A . Previous upper bounds of this nature on $D(A)$ include:

- $D(A) \leq (N^0(A) + 1)(N^1(A) + 1)$.
- $D(A) \leq \log \text{rank}(A)(N^0(A) + 1)$.
- $D(A) \leq \log \text{rank}(-A)(N^1(A) + 1)$.
- $D(A) \leq \log \text{rank}(A) \log \text{rank}_+(A)$.

Here, $N^1(A)$ and $N^0(A)$ are the nondeterministic and co-nondeterministic communication complexity of A , respectively. Also, $\text{rank}(A)$ is the rank of A , and $\text{rank}_+(A)$ is the positive rank of A . All these complexity measures are known lower bounds on the deterministic communication complexity. See [9] or [12] for a comprehensive survey of these complexity measures and bounds.

Recently, Gavinsky and Lovett [5] augmented this repertoire with additional bounds. They proved that the deterministic communication complexity of a sign matrix A is at most $O(CC(A) \log^2 \text{rank}(A))$, where $CC(A)$ is either randomized communication complexity, information complexity, or zero-communication complexity. Thus when the rank of the matrix is low, an efficient nondeterministic protocol or a randomized protocol, implies an efficient deterministic protocol.

The heart of the proofs in [5] is a clever and simple lemma stating that when the fraction of either 1's or -1 's in a matrix is small compared to the rank of the matrix then it contains a large monochromatic rectangle. We use this lemma to prove the following bounds:

$$D(A) \leq O(\text{corr}_{1/4}^{(1)}(A) \log^2 \text{rank}(A)),$$

and

$$D(A) \leq O(\text{corr}_{1/4}^{(1)}(-A) \log^2 \text{rank}(A)),$$

where $\text{corr}_{1/4}^{(1)}(A)$ is a one-sided version of the corruption/rectangle bound with error $1/4$. That is, the corruption bound is the maximum of $\text{corr}_{1/4}^{(1)}(A)$ and $\text{corr}_{1/4}^{(1)}(-A)$.

The corruption bound is smaller than the randomized complexity, information complexity, and zero-communication complexity (see e.g. [7]). Therefore our bounds are smaller than the bounds proved in [5]. The one-sided version of the corruption bound, $\text{corr}_{1/4}^{(1)}(A)$, is smaller than the nondeterministic communication complexity. Similarly, $\text{corr}_{1/4}^{(1)}(-A)$ is smaller than the co-nondeterministic communication complexity. Thus, our bounds unify the two groups of previous bounds mentioned above, and fall very naturally in this framework.

In fact, since MA -complexity is bounded by root of the corruption bound [8], we get as a consequence that

$$D(A) \leq O(MA(A)^2 \log^2 \text{rank}(A)),$$

and

$$D(A) \leq O(MA(-A)^2 \log^2 \text{rank}(A)).$$

Here $MA(A)$ is the Merlin Arthur (MA) complexity of A , with error $1/4$. In this model the players first make a nondeterministic guess and then perform a randomized protocol. This has the nice interpretation that when the rank is low, there is an efficient deterministic protocol, even compared with protocols combining the power of nondeterminism and randomization.

The corruption bound (and variants of it) is a central lower bound technique for the randomized communication complexity. It was used already by Yao in [18], and later e.g. in [1, 17]. It is the only lower bound technique (excluding the relatively new partition bound [6]), that has not been transferred also to quantum communication. Log of the rank, on the other hand, is proved to also bound exact quantum communication complexity [3], even with entanglement [4]. Therefore, the upper bounds in terms of corruption and log-rank imply that the corruption bound is a lower bound on exact quantum communication complexity, if and only if quantum communication is equivalent to deterministic communication complexity.

We give the relevant definitions and background in Section 2, and prove the upper bounds in Section 3. In Section 4 we give a simple proof to the novel bound on the deterministic communication complexity in terms of the rank, proved by Lovett [15].

2 Definitions

Let A be an $m \times n$ sign matrix, and let μ be a probability distribution on $[m] \times [n]$. For a set of entries $I \subseteq [m] \times [n]$, let $\mu(I)$ be the sum $\sum_{(i,j) \in I} \mu(i,j)$. A combinatorial rectangle is a subset $S \times T$ of entries, such that $S \subseteq [m]$ and $T \subseteq [n]$. With a slight abuse of notation, for $v \in \{\pm 1\}$ and a combinatorial rectangle R , we denote by $\mu(v) = \mu(\{(i,j) | A_{i,j} = v\})$, and $\mu(v, R) = \mu(\{(i,j) \in R | A_{i,j} = v\})$. We also write $\mu(v|R)$ for the conditional probability equal to $\mu(v, R)/\mu(R)$. Finally, let

$$\text{err}(R, \mu, v) = \mu(-v|R).$$

Intuitively, $\text{err}(R, \mu, v)$ represents the probability of error in the rectangle R , assuming a randomized protocol answers v on R .

Definition 1 Let A be a sign matrix, $0 \leq \epsilon \leq 1$, and $v \in \{-1, 1\}$. Define

$$\text{size}_\epsilon^{(v)}(A, \mu) = \max_R \{\mu(R) : \text{err}(R, \mu, v) \leq \epsilon\},$$

where the maximum is over all combinatorial rectangles R .

Usually, when defining the corruption bound, balanced distributions are considered. That is, distributions for which the probability of -1 and the probability of 1 are not too small. Here, we consider strictly balanced distributions that have their support on a combinatorial rectangle. We also add the restriction that the distributions are uniform on the set of entries equal to -1 , and also on the set of entries equal to 1 .

Formally, we say that a distribution μ on $[m] \times [n]$ is uniformly-balanced, if it satisfies:

- The set of entries (i, j) for which $\mu(i, j) > 0$ is a combinatorial rectangle.
- $A_{i,j} = A_{x,y}$ implies that $\mu(i, j) = \mu(x, y)$, if both are nonzero.
- $\mu(1) = \mu(-1) = \frac{1}{2}$.

The corruption bound is:

Definition 2 Let A be a sign matrix, $0 \leq \epsilon \leq 1$, and $v \in \{-1, 1\}$. Define

$$\text{corr}_\epsilon^{(v)}(A) = \max_\mu \log 1/\text{size}_\epsilon^{(v)}(A, \mu),$$

where μ runs over all uniformly-balanced distributions. Also define

$$\text{corr}_\epsilon(A) = \max_{v \in \{-1, 1\}} \{\text{corr}_\epsilon^{(v)}(A)\}.$$

When A is monochromatic, define $\text{corr}_\epsilon(A) = 0$. If A is all 1 's then define $\text{corr}_\epsilon^{(1)}(A) = 0$, and otherwise define $\text{corr}_\epsilon^{(-1)}(A) = 0$. Note that when A is monochromatic then there are no balanced distributions on A , thus an explicit definition is required for this special case.

Note that the above version of the corruption bound is smaller than the usual definition, since we consider only a subset of the distributions. But, as we are interested in proving an upper bound, this works to our advantage, as long as we can still prove the properties we need.

3 The upper bound

We prove the following upper bound:

Theorem 3 Let A be a sign matrix. Then

$$D(A) \leq O(\text{corr}_{1/4}^{(1)}(A) \log^2 \text{rank}(A)),$$

and

$$D(A) \leq O(\text{corr}_{1/4}^{(-1)}(A) \log^2 \text{rank}(A)).$$

Observe that, $\text{corr}_{1/4}^{(-1)}(A) = \text{corr}_{1/4}^{(1)}(-A)$.

The proof of the upper bound is composed of three main ingredients:

- An amplification lemma for the error in the corruption bound, [8].

- A lemma relating the size of monochromatic rectangles to the size of corrupted rectangles with small error (relative to the rank), [5, 15].
- Nisan and Wigderson's deterministic protocol, [16]. This protocol uses the existence of a large monochromatic rectangle.

A sketch of the proof is: If the corruption bound is small, then for every probability distribution there is a large rectangle with a small fraction of either 1's or -1's, by definition. Using the amplification lemma, this fraction can be made small even relative to the rank. Then, by the second part, there is a large monochromatic rectangle, allowing the use of the protocol of Nisan and Wigderson.

In combining these three parts together, a few modifications and adjustments are required, most of which are contained in Corollary 6 relating the corruption bound to the size of a monochromatic rectangle.

We start with the amplification lemma for the corruption bound.

Lemma 4 ([8]) *Let A be a sign matrix, $0 \leq \epsilon < 1/2$ ¹, and $\ell \in \mathbb{N}$. Then*

$$\text{corr}_{\epsilon^\ell}^{(1)}(A) \leq O(\ell \cdot \text{corr}_\epsilon^{(1)}(A)).$$

For convenience, we sketch the proof. We also observe that the proof in [8] gives more than revealed in the statement of the lemma².

Proof [Sketch] Let $k = \text{corr}_\epsilon^{(1)}(A)$, and fix any uniformly-balanced distribution μ on the entries of A . Then, there is a rectangle R of measure at least 2^{-k} and with at most an ϵ -fraction of -1's, with respect to μ . Now, let μ_1 be the uniformly-balanced distribution supported on R . Again, there is a rectangle R_1 in R with measure at least 2^{-k} and at most an ϵ -fraction of -1's, with respect to μ_1 . The measure of R_1 with respect to μ is at least 2^{-2k} , and the fraction of -1's is at most $O(\epsilon^2)$ (see [8] for the details). Repeating this process gives the proof.

Intuitively, picking the new distribution μ_1 allows to "hide" the small fraction of -1's, and search for a smaller fraction inside this set. It is left to observe that when the starting distribution μ is uniformly-balanced, the distribution μ_1 picked in the process is also uniformly balanced. Thus, the proof in [8] is valid for our restricted version of the corruption bound. ■

The following lemma implies that amplification achieves a monochromatic rectangle when the error is proportional to $1/\text{rank}(A)$.

Claim 5 ([5, 15]) *Let A be an $m \times n$ sign matrix with $\text{rank}(A) = r$. Assume that the fraction of 1's or the fraction of -1's in A is at most $\frac{1}{2r}$. Then A contains a monochromatic rectangle R such that $|R| \geq \frac{mn}{8}$.*

We note that the proof of Claim 5 is purely algebraic, and thus it holds for the rank over any field. In particular, it holds with rank over \mathbb{F}_2 which can be much smaller than the rank over the reals. This is worth noting since the log rank conjecture is false for the rank over \mathbb{F}_2 , e.g. the Inner Product (Hadamard) matrix. Thus, in any proof of the log rank conjecture, at least one ingredient must use inimitable properties of the rank over \mathbb{R} .

Lemma 4 and Claim 5, imply the following corollary:

Corollary 6 *Let A be a sign matrix, $q_1 = \log \text{rank}(A) \cdot \text{corr}_{1/4}^{(1)}(A)$, $q_2 = \log \text{rank}(A) \cdot \text{corr}_{1/4}^{(1)}(-A)$, and also let $q = \min\{q_1, q_2\}$. Then A contains a monochromatic rectangle of normalized size $2^{-O(q)}$.*

¹Here ϵ must be bounded away from $1/2$. A more precise lemma, dealing also with the case of ϵ close to $1/2$, is stated in Section 4.

²We provide a complete and independent proof for a more general lemma in Section 5.

Proof Let $\text{rank}(A) = r$, and assume without loss of generality that $q_1 \leq q_2$. If the fraction of either -1 's or 1 's in A is smaller than $1/2r$, we can apply Claim 5 and conclude. Otherwise, let μ be the uniformly-balanced distribution supported on all the entries of A , and denote by u the uniform distribution on the entries of A . Let $\delta = u(-1)$, be the fraction of -1 's in A . Then for every entry (i, j) it holds that

$$\begin{aligned} A_{i,j} = -1 &\Rightarrow \mu(i, j) = \frac{u(i, j)}{2\delta}, \\ A_{i,j} = 1 &\Rightarrow \mu(i, j) = \frac{u(i, j)}{2(1-\delta)}. \end{aligned} \quad (1)$$

Let $\text{corr}_{1/4r}^{(1)}(A) = k$. Since μ is uniformly-balanced, there is a rectangle R such that $\mu(R) \geq 2^{-k}$ and $\text{err}(R, \mu, 1) \leq 1/4r$.

By Equation (1) and the fact that $1/2r \leq \delta \leq 1 - 1/2r$, we have that

$$u(R) \geq \frac{1}{r} 2^{-k} = 2^{-k - \log r}.$$

In addition, $\mu(-1, R) \leq 1/4r(\mu(-1, R) + \mu(1, R))$. By (1) again, the latter inequality gives

$$\frac{u(-1, R)}{2\delta} \leq \frac{1}{4r} \left(\frac{u(-1, R)}{2\delta} + \frac{u(1, R)}{2(1-\delta)} \right).$$

Assume $\delta \leq 1/2$, then multiplying both sides by 2δ yields,

$$\begin{aligned} u(-1, R) &\leq \frac{1}{4r} \left(u(-1, R) + \frac{\delta u(1, R)}{(1-\delta)} \right) \\ &\leq \frac{1}{4r} (u(-1, R) + u(1, R)) \\ &= \frac{1}{4r} u(R). \end{aligned}$$

If $\delta > 1/2$ then

$$u(-1, R) \leq 2\mu(-1, R) \leq \mu(R)/2r \leq u(R)/2,$$

and we can repeat the argument on R . This decreases the size of the rectangle that we find by at most an extra factor of 2^{-k} .

In both cases we therefore find a combinatorial rectangle R of normalized size at least $2^{-2k - \log r}$, and a fraction of -1 's at most $1/4r$. Now, by Claim 5, there exists a monochromatic rectangle R' such that

$$|R'| \geq |R|/8 \geq 2^{-2k - \log r - 3}.$$

To conclude the proof, apply Lemma 4 with $\ell = \lceil \log_4 4r \rceil$, which gives

$$\text{corr}_{1/4r}^{(1)}(A) \leq O(\text{corr}_{1/4}^{(1)}(A) \log r).$$

■

It is left to state the result of Nisan and Wigderson [16]. We use a similar formulation to that used in [5, 15]. Though the statement is slightly different than that of [16], the proof remains the same.

Claim 7 ([16, 5, 15]) *Let A be a sign matrix and let $\text{rank}(A) = r$. Assume that for every sub-matrix of A there exists a monochromatic rectangle of normalized size at least 2^{-q} . Then,*

$$D(A) \leq O(\log^2 r + q \log r).$$

Here too we note that the use of the rank in the proof is algebraic. Thus the claim holds with the rank over any field.

For the proof of Theorem 3 we combine Corollary 6, Claim 7 and the fact that the corruption bound is monotone in sub-matrices.

Proof [Theorem 3] Let $\text{rank}(A) = r$. We show that

$$D(A) \leq O(\text{corr}_{1/4}^{(1)}(A) \log^2 r).$$

For every sub-matrix B of A and every $0 \leq \epsilon \leq 1$, $\text{corr}_\epsilon^{(1)}(B) \leq \text{corr}_\epsilon^{(1)}(A)$. Let $q = \text{corr}_{1/4}^{(1)}(A) \log r$. Then by Corollary 6, every sub-matrix of A contains a monochromatic rectangle of normalized size $2^{-O(q)}$. The proof now follows using Claim 7.

The proof that $D(A) \leq O(\text{corr}_{1/4}^{(-1)}(A) \log^2 r)$ is similar. In fact, it also follows by symmetry, since $D(A) = D(-A)$ and $\text{corr}_{1/4}^{(-1)}(A) = \text{corr}_{1/4}^{(1)}(-A)$. \blacksquare

4 On the log rank conjecture

As mentioned in the introduction, upper bounds in communication complexity are particularly interesting when stated in terms of known lower bounds, as this gives an alternative characterization of the underlying communication complexity measure. A fundamental question in communication complexity, posed by Lovász and Saks [13, 14], is whether deterministic communication complexity is bounded by a polynomial in the log of the rank of the matrix.

There is a simple deterministic protocol using $\text{rank}(A)$ bits of communication. Despite significant efforts, no better bounds were found until recently. A breakthrough result of Lovett [15] (following also [5] and [2]) is the first improvement on this simple upper bound. We give a simple proof of this result using the framework described in the previous sections. The proof is essentially observing that discrepancy corresponds to the error in corruption, and the error can be amplified as seen in Section 3. We therefore prove the following:

Lemma 8 *Let A be a sign matrix, and let $\text{disc}(A) = 1/d$. Then*

$$\text{corr}_{1/4}(A) \leq O(d \log d).$$

Here $\text{disc}(A)$ is the discrepancy of A , defined as follows: Let μ be a distribution on the entries of A . The discrepancy with respect to μ is the maximal discrepancy between the measure of 1's and the measure of -1's, over combinatorial rectangles in A . The discrepancy of A is the minimal discrepancy over all probability distributions.

The discrepancy is often used to lower bound communication complexity in different models, and it is also equivalent (up to a constant) to the reciprocal of margin complexity. See [10, 11] for the definitions and proof of the equivalence of these measures. This equivalence was used in [11] to prove that $1/\text{disc}(A) \leq O(\sqrt{\text{rank}(A)})$. Combined with Lemma 8 it implies that $\text{corr}_{1/4}(A) \leq O(\sqrt{\text{rank}(A)} \log \text{rank}(A))$. Using this in Theorem 3 gives the upper bound of [15] on the deterministic communication complexity in terms of the rank (up to log factors). We prove Lemma 8 in the remainder of this section.

We use the following lemma, generalizing the amplification lemma of [8]³.

Lemma 9 *Let A be a sign matrix. Then, for $1/4 < \epsilon < 1/2$ it holds that*

$$\text{corr}_{1/4}^{(1)}(A) \leq O\left(\frac{1}{(1/2 - \epsilon)} \cdot \text{corr}_\epsilon^{(1)}(A)\right),$$

and, for $0 \leq \epsilon \leq 1/4$ and $\ell \in \mathbb{N}$

$$\text{corr}_{\epsilon^\ell}^{(1)}(A) \leq O(\ell \cdot \text{corr}_\epsilon^{(1)}(A)).$$

³Studying the proof of Lemma 4 in [8], it can be adapted to give a proof for this generalized lemma. We give an independent proof in Section 5.

We use the above lemma to amplify the discrepancy between the 1's and -1 's in A in order to get the bound on the corruption bound: By definition of the discrepancy, for every probability distribution μ on the entries of A , there is a combinatorial rectangle R , such that

$$\left| \sum_{(i,j) \in R} \mu(i,j) A_{i,j} \right| \geq \frac{1}{d}. \quad (2)$$

Thus, for every probability distribution there is a combinatorial rectangle for which the probability of either 1's or -1 's in it is at most $1/2 - 1/2d$. We show that this implies that $\text{corr}_{1/2-1/6d}^{(1)}(A) = O(\log d)$. Proving similarly for $\text{corr}_{1/2-1/6d}^{(-1)}(A)$ and applying Lemma 9 gives the bound $\text{corr}_{1/4}(A) \leq O(d \log d)$.

It is left to prove the bound on $\text{corr}_{1/2-1/6d}^{(1)}(A)$. Let μ be a uniformly-balanced distribution. We first observe that we can assume without loss of generality that

$$\sum_{(i,j) \in R} \mu(i,j) A_{i,j} \geq \frac{1}{3d}.$$

Otherwise, the sum in Equation (3) is negative, and since μ is uniformly-balanced

$$\sum_{(i,j) \in \bar{R}} \mu(i,j) A_{i,j} \geq \frac{1}{d},$$

where \bar{R} is the complement of R . But \bar{R} can be partitioned into three combinatorial rectangles, and thus there is a rectangle R' such that

$$\sum_{(i,j) \in R'} \mu(i,j) A_{i,j} \geq \frac{1}{3d}.$$

Now, $\sum_{(i,j) \in R} \mu(i,j) A_{i,j} = \mu(1, R) - \mu(-1, R)$, and $\mu(R) = \mu(1, R) + \mu(-1, R)$. Therefore,

$$\begin{aligned} \mu(-1, R) &= \frac{1}{2}\mu(R) - \frac{1}{2} \sum_{(i,j) \in R} \mu(i,j) A_{i,j} \\ &\leq \frac{1}{2}\mu(R) - \frac{1}{6d} \\ &= \frac{1}{2}\mu(R) - \frac{\mu(R)}{6d\mu(R)} \\ &= \left(\frac{1}{2} - \frac{1}{6d\mu(R)}\right)\mu(R) \\ &\leq \left(\frac{1}{2} - \frac{1}{6d}\right)\mu(R). \end{aligned}$$

This concludes the proof, as obviously $\mu(R) \geq \frac{1}{d}$.

5 Proof of Lemma 9

Lemma 10 (Restated) *Let A be a sign matrix. Then, for $1/4 < \epsilon < 1/2$ it holds that*

$$\text{corr}_{1/4}^{(1)}(A) \leq O\left(\frac{1}{(1/2 - \epsilon)} \cdot \text{corr}_{\epsilon}^{(1)}(A)\right),$$

and, for $0 \leq \epsilon \leq 1/4$ and $\ell \in \mathbb{N}$

$$\text{corr}_{\epsilon^\ell}^{(1)}(A) \leq O(\ell \cdot \text{corr}_{\epsilon}^{(1)}(A)).$$

Note that a unified way to state the lemma is: for every $0 < \epsilon_1 < \epsilon < 1/2$ it holds that

$$\text{corr}_{\epsilon_1}^{(1)}(A) \leq O(\log_{\epsilon} \epsilon_1 \cdot \text{corr}_{\epsilon}^{(1)}(A)).$$

Proof

We start with a single step of refinement. Let $\text{corr}_{\epsilon}^{(1)}(A) = k$, and pick a uniformly-balanced distribution μ . By definition there is a rectangle R such that $\mu(R) \geq 2^{-k}$ and $\text{err}(R, \mu, 1) \leq \epsilon$, we show that there is a rectangle R_1 such that $\mu(R_1) \geq 2^{-2k}$ and $\text{err}(R_1, \mu, 1) \leq 2\epsilon^2$.

Let μ_1 be the uniformly balanced distribution on R . Then for every entry (i, j) it holds that

$$\begin{aligned} A_{i,j} = -1, (i, j) \in R &\Rightarrow \mu_1(i, j) = \frac{\mu(i, j)}{2\mu(-1, R)}, \\ A_{i,j} = 1, (i, j) \in R &\Rightarrow \mu_1(i, j) = \frac{\mu(i, j)}{2(\mu(R) - \mu(-1, R))}. \end{aligned} \quad (3)$$

Since μ_1 is uniformly-balanced, there is a rectangle R_1 such that $\mu_1(R_1) \geq 2^{-k}$ and $\text{err}(R_1, \mu_1, 1) \leq \epsilon$. By Equation (3):

$$\begin{aligned} \mu(R_1) &= \sum_{(i,j) \in R_1} \mu(i, j) \\ &= \sum_{A_{i,j}=-1, (i,j) \in R_1} \mu(i, j) + \sum_{A_{i,j}=1, (i,j) \in R_1} \mu(i, j) \\ &= \sum_{A_{i,j}=-1, (i,j) \in R_1} 2\mu(-1, R)\mu_1(i, j) + \sum_{A_{i,j}=1, (i,j) \in R_1} 2(\mu(R) - \mu(-1, R))\mu_1(i, j) \\ &= 2\mu(-1, R) \sum_{A_{i,j}=-1, (i,j) \in R_1} \mu_1(i, j) + 2(\mu(R) - \mu(-1, R)) \sum_{A_{i,j}=1, (i,j) \in R_1} \mu_1(i, j) \\ &= 2\mu(-1, R)\mu_1(-1, R_1) + 2(\mu(R) - \mu(-1, R))(\mu_1(R_1) - \mu_1(-1, R_1)) \\ &= 2[\mu(-1|R)\mu(R)\mu_1(-1|R_1)\mu(R_1) + (\mu(R) - \mu(-1|R)\mu(R))(\mu_1(R_1) - \mu_1(-1|R_1)\mu(R_1))] \\ &= 2\mu(R)\mu(R_1)[\mu(-1|R)\mu_1(-1|R_1) + (1 - \mu(-1|R))(1 - \mu_1(-1|R_1))]. \end{aligned}$$

Recall that $\mu(-1|R) \leq \epsilon < 1/2$ and $\mu_1(-1|R_1) \leq \epsilon < 1/2$. The function $f(x, y) = 1 - x - y + 2xy$ satisfies $f(x, y) \geq 1/2$ for $x, y \in [0, 1/2]$. Thus

$$\begin{aligned} \mu(R_1) &= 2\mu(R)\mu(R_1)[\mu(-1|R)\mu_1(-1|R_1) + (1 - \mu(-1|R))(1 - \mu_1(-1|R_1))] \\ &\geq \mu(R)\mu_1(R_1) \\ &\geq 2^{-2k}. \end{aligned}$$

It is left to consider the error $\text{err}(R_i, \mu, 1)$, for $1 \leq i \leq T$. Again we start with a single step, recall that $\text{err}(R_1, \mu, 1) = \mu(-1|R_1) = \mu(-1, R_1)/\mu(R_1)$. Using Equation (3) we get

$$\begin{aligned} \mu(-1, R_1) &= \sum_{A_{i,j}=-1, (i,j) \in R_1} \mu(i, j) \\ &= \sum_{A_{i,j}=-1, (i,j) \in R_1} 2\mu(-1, R)\mu_1(i, j) \\ &= 2\mu(-1, R) \sum_{A_{i,j}=-1, (i,j) \in R_1} \mu_1(i, j) \\ &= 2\mu(-1, R)\mu_1(-1, R_1) \\ &\leq 2\epsilon^2\mu(R)\mu(R_1). \end{aligned}$$

Combining the last inequality with the inequality $\mu(R_1) \geq \mu(R)\mu_1(R_1)$ proved earlier, we get

$$\begin{aligned}\mu(-1|R_1) &= \mu(-1, R_1)/\mu(R_1) \\ &\leq \frac{2\epsilon^2\mu(R)\mu(R_1)}{2\mu(R)\mu_1(R_1)} \\ &= 2\epsilon^2.\end{aligned}$$

Similarly, after T steps of refinement, finding finer and finer rectangles R_1, R_2, \dots, R_T , we get $\mu(R_T) \geq 2^{-(T+1)k}$, and the error satisfies $\mu(-1|R_T) \leq 2^T\epsilon^{T+1}$.

Therefore, for $\epsilon \leq 1/4$, after $T = 2l$ steps we achieve error at most $2^{2l}\epsilon^{2l+1} \leq \epsilon^l$. For $1/4 < \epsilon < 1/2$ let $\delta = 1/2 - \epsilon$. After T steps the error is at most $(1 - 2\delta)^T$, and for $T = c/\delta$ steps the error is smaller than $1/4$, for some constant c . ■

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